

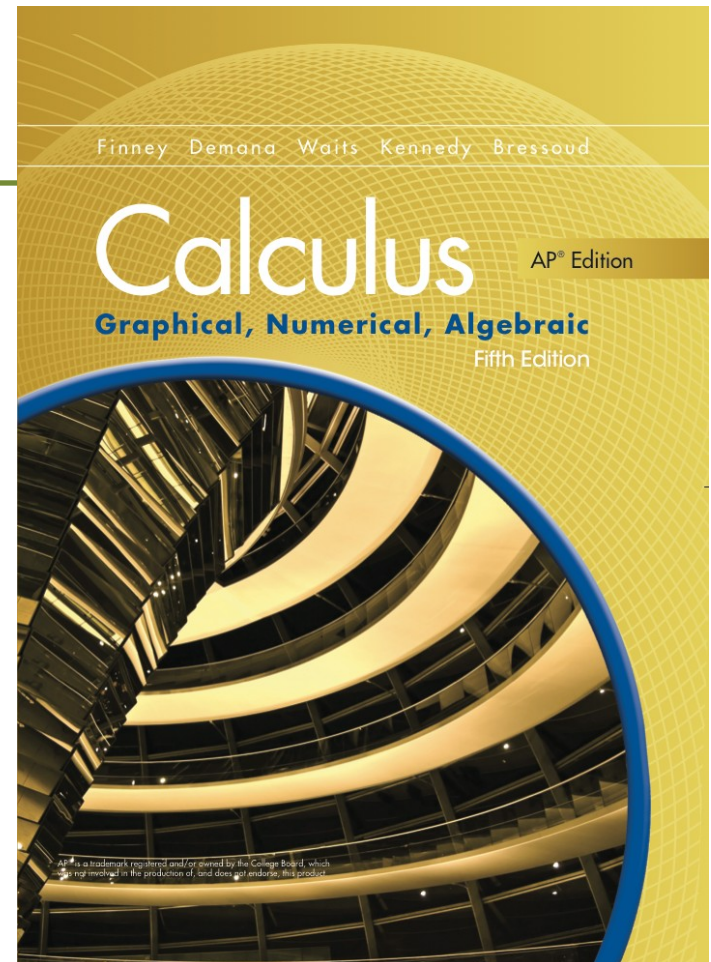
Chapter 10

Infinite Series



Section 10.4

Radius of Convergence



Quick Review

Find the limit of the expression as $n \rightarrow \infty$. Assume x remains fixed as n changes.

1. $\frac{nx}{n+1}$

2. $\frac{n^2|x-3|}{n(n-1)}$

3. $\frac{|x|}{n!}$

4. $\frac{(n+1)^4 x^2}{(2n)^4}$

5. $\frac{2^n |2x+1|^{n+1}}{2^{n+1} |2x+1|^n}$

Quick Review

Let a_n be the n th term of the first series and b_n the n th term of the second series. Find the smallest positive integer N for which $a_n > b_n$ for all $n \geq N$. Identify a_n and b_n .

6. $\sum 5n, \quad \sum n^2$

7. $\sum n^5, \quad \sum 5^n$

8. $\sum \ln n, \quad \sum \sqrt{n}$

9. $\sum \frac{1}{10^n}, \quad \sum n^3$

Quick Review Solutions

Find the limit of the expression as $n \rightarrow \infty$. Assume x remains fixed as n changes.

$$1. \frac{n|x|}{n+1} \quad |x|$$

$$2. \frac{n^2|x-3|}{n(n-1)} \quad |x-3|$$

$$3. \frac{|x|}{n!} \quad 0$$

$$4. \frac{(n+1)^4 x^2}{(2n)^4} \quad x^2/16$$

$$5. \frac{2^n |2x+1|^{n+1}}{2^{n+1} |2x+1|^n} \quad |2x+1|/2$$

Quick Review Solutions

Let a_n be the n th term of the first series and b_n the n th term of the second series. Find the smallest positive integer N for which $a_n > b_n$ for all $n \geq N$. Identify a_n and b_n .

6. $\sum 5n, \sum n^2$ $a_n = n^2, b_n = 5n, N=6$

7. $\sum n^5, \sum 5^n$ $a_n = 5^n, b_n = n^5, N=6$

8. $\sum \ln n, \sum \sqrt{n}$ $a_n = \sqrt{n}, b_n = \ln n, N=1$

9. $\sum \frac{1}{10^n}, \sum \frac{1}{n!}$ $a_n = \frac{1}{10^n}, b_n = \frac{1}{n!}, N=25$

10. $\sum \frac{1}{n^2}, \sum n^3$ $a_n = \frac{1}{n^2}, b_n = n^3, N=2$

What you'll learn about

- The importance of convergence
- The n th-Term Test
- The Direct Comparison Test
- Absolute and conditional convergence
- The Ratio Test
- The radius of convergence of a power series
- Possible behavior at the endpoints of an interval of convergence
- Telescoping series

What you'll learn about

... and why

It is important to develop a strategy for finding the

interval of convergence of a power series and to obtain

some tests that can be used to determine convergence of a series.

The Convergence Theorem for Power Series

There are three possibilities for $\sum_{n=0}^{\infty} c_n (x - a)^n$ with respect to convergence:

1. There is a positive number R such that the series diverges for $|x - a| > R$ but converges for $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

The n th-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \rightarrow \infty} a_n$ fails to exist or is different from zero.

The Direct Comparison Test

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N .

(b) $\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N .

Example Proving Convergence by Comparison

Prove that $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$ converges for all real x .

Let x be any real number. The series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$ has no negative terms.

For any n , $\frac{x^{2n}}{(n!)^2} \leq \frac{x^{2n}}{n!} = \frac{(x^2)^n}{n!}$. Recognize $\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$ as the Taylor series

for e^{x^2} , which we know converges to e^{x^2} for all real numbers. Since

the e^{x^2} series dominates $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$ term by term, the latter series must also

converge for all real numbers by the Direct Comparison Test.

Absolute Convergence

If the series $\sum |a_n|$ of absolute values converges, then $\sum a_n$ **converges absolutely**.

Absolute Convergence Implies Convergence

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Example Using Absolute Convergence

Show that $\sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$ converges for all x

Let x be any real number. The series $\sum_{n=0}^{\infty} \frac{|\sin x|^n}{n!}$ has no negative terms,

and it is term-by-term less than or equal to the series $\sum_{n=0}^{\infty} \frac{1}{n!}$, which converges

to e . Therefore, $\sum_{n=0}^{\infty} \frac{|\sin x|^n}{n!}$ converges by direct comparison.

Since $\sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$ converges absolutely, it converges.

The Ratio Test

Let $\sum a_n$ be a series with positive terms, and with $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$.

Then,

- (a)** the series converges if $L < 1$,
- (b)** the series diverges if $L > 1$,
- (c)** the test is inconclusive if $L = 1$.

Example Finding the Radius of Convergence

Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{nx^n}{10^n}$.

Check for absolute convergence using the Ratio Test.

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{(n+1)|x^{n+1}|}{10^{n+1}} \cdot \frac{10^n}{n|x^n|} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) \frac{|x|}{10} = \frac{|x|}{10}\end{aligned}$$

Setting $\frac{|x|}{10} < 1$, we see that the series converges absolutely for $-10 < x < 10$.

The series diverges for $x > 10$ and $x < -10$.

The radius of convergence is 10.

Example Determining Convergence of a Series

Determine the convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{2^n}{3^n + 1}$.

Use the Ratio Test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{3^{n+1} + 1}}{\frac{2^n}{3^n + 1}} = \lim_{n \rightarrow \infty} \left(\frac{2^{n+1}}{3^{n+1} + 1} \right) \left(\frac{3^n + 1}{2^n} \right) \\ &= \lim_{n \rightarrow \infty} 2 \left(\frac{3^n + 1}{3^{n+1} + 1} \right) \\ &= \lim_{n \rightarrow \infty} 2 \frac{1 + \frac{1}{3^n}}{3 + \frac{1}{3^n}} \\ &= \frac{2}{3} \quad \text{The series converges because } \frac{2}{3} < 1.\end{aligned}$$